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Purity and Pure-Injectivity for Topological Modules

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Dedicated to Professor Mihály Makkai on the occasion of his seventieth birthday

Abstract. A logic suitable for topological structures was introduced by McKee and Sgro in the 1970s and developed further by Garavaglia, Ziegler and Kucera. This logic satisfies the compactness theorem, a Löwenheim – Skolem theorem, and a Lindström theorem.

The model theory of modules is very rich with strong connections to standard questions of algebra, primarily based on the partial elimination of quantifiers down to positive primitive formulas. Theories of topological modules also have a partial elimination of quantifiers, down to "topological positive primitive formulas," so there is some hope to see if a development of these theories can be made that parallels the development of the model theory of ordinary modules.

In his M.Sc. thesis, the first author investigated whether there are good definitions of "pure embedding" and of "pure-injective module" for topological modules in this context. The answer is "mostly not." We report on these results and related results from the literature here.

Topology is innately a higher order subject: statements about topological properties involve quantifications over individuals, over open sets, and over families of open sets. Nonetheless, in the mid to late 1970s, several groups of people realized independently that many concepts of topological interest could be expressed in a first-order context. The language $\mathcal{L}_{\mathrm{t}}$ was introduced by McKee [21–23] and Sgro [30]; and Garavaglia developed these ideas much further [10]. Meanwhile Martin Ziegler had also been studying similar problems [31], and, with Jörg Flum, published a comprehensive overview of the subject in 1980 [9]. The second author developed some of these ideas further in his Ph.D. thesis [16], in particular aspects of stability theory and applications to theories of topological modules [17]. But in spite of limited attempts in [17] and later efforts by Majewski [20], no really

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This paper deals in part with problems left unresolved in the Ph.D. thesis of the second author, prepared under the direction of Dr. Makkai, and submitted to McGill University in 1984.

This is the final form of the paper.

satisfactory formulations of the concepts of "elementary extension" or "saturated model" have been made. In particular, neither author offered any examples.

An extensive list of the simpler topological properties can be axiomatized in the language $\mathcal{L}_{\mathrm{t}}$, but in spite of all this, it is quite limited in its expressive power, and the question arises as to what sorts of "everyday" mathematics can be done in this context. That is the question that concerns us here; in particular, we ask "Is there an interesting and/or useful model theory for topological modules (parallel to the very successful model theory of ordinary modules) that can be developed in $\mathcal{L}_{\mathrm{t}}$?"

The answer is "Almost certainly not."

A question like this can't be answered definitively by some sort of incompleteness theorem, or by specific counterexamples, but only by the cumulative weight of evidence. We show that a certain set of core ideas in the model theory of ordinary modules cannot be expected to generalize to topological modules in any natural way. Our investigation of this question is completely tied to the expressive power of the languages $\mathcal{L}_{\mathrm{t}}$ and $\mathcal{L}_{\mathrm{m}}$; and we do not rule out the possibility that alternative approaches might be more profitable. We make some comments about these approaches at the end.

Part of our motivation is this: it is easier for an embedding of ordinary modules to be pure than it is for it to be elementary; it is easier for a module to be pure-injective than it is for it to be saturated. Thus exploring the nature or feasibility of these concepts for topological modules may shed light on the problems associated with defining the more general concepts for arbitrary theories in $\mathcal{L}_{\mathrm{t}}$, but in a context where we might have more powerful tools at our disposal.

Convention A. Structures for some first-order language are denoted by a script letter $\mathcal{M}, \mathcal{A}, \mathcal{M}_0$, etc; and their underlying sets are consistently denoted by the corresponding uppercase Roman letter M, A, M_0, etc.

Convention B. $\bar{v}, \overline{m}$ ($\overline{m} \in M$), $\overline{U}$, etc., represent finite tuples of variables, individuals (in M), sets, etc. Operations (such as $f(\overline{m}), \overline{U} \cap M$) applied to a tuple are to be interpreted as applying componentwise.

1. Model theory of modules

What are some of the basic ideas that make the model theory of modules so successful? It is these ideas for which we will want to develop parallels in the model theory of topological modules, if possible.

Definition 1.1. Let R be a ring with unit. The *language* ${}_R\mathcal{L}$ *of* R*-modules* has the symbols $+, -$, and $\mathbf{0}$ of the usual language of abelian groups, as well as for each $r \in R$ a unary operation symbol $\mathbf{r}$, intended to represent scalar multiplication by r.

It is easy to see how to axiomatize "left (or right) module over R" in this language; cf. Eklof and Sabbagh [7]. Building the ring into the language rather than axiomatizing R-modules as two-sorted structures avoids including the already known-to-be-complicated theory of rings into the study of the model theory of modules. Since the ring in question is seldom important for our arguments, we rarely mention it. All results apply equally to left and to right R-modules.

Convention C. All modules are left modules over some fixed ring R, unless otherwise stated. We do not distinguish between abelian groups and (left) $\mathbb{Z}$-modules.

Using this language, a very rich and fruitful model theory of modules has been developed. For all basic definitions and references to the early history of the subject, we refer the reader to Prest's book [28]. A good sense of how the subject area has developed and become entwined with contemporary algebra — and in many ways, has separated from its roots in model theory — over the last 30 years can be obtained from Prest's recent book [29].

Definition 1.2. A *positive primitive formula* (pp formula) is a formula of the form $\exists w_1 \ldots \exists w_n \bigwedge_{j<m} \alpha_j$, where $n \geq 0$, $m \geq 1$, and each α_j is an atomic formula.

Note that up to logical equivalence, the set of positive primitive formulas is closed under existential quantification and conjunctions. For languages and theories of modules, we may assume that each atomic formula is of the form $\mathbf{t} = \mathbf{0}$ for some term $\mathbf{t}$ of ${}_R\mathcal{L}$, specifically of the form $\mathbf{t} = \mathbf{r}_1 u_1 + \cdots + \mathbf{r}_k u_k$ for distinct variables $u_1, \ldots, u_k$.

Here are some of the basic properties of pp formulas in modules; these may all be found in Prest [28, Chapter 2; 2.1, 2.2, 2.7, 2.10].

Proposition 1.3. *Let $\mathcal{M}$ and $\mathcal{N}$ be left R-modules, $f\colon \mathcal{M} \to \mathcal{N}$ an R-module homomorphism, and φ and ψ pp formulas with the same free variables, say $v_0, \ldots, v_{n-1}$.*

(1) "*pp formulas define subgroups*": *$\varphi[M] = \{\overline{m} \in M^n : \mathcal{M} \models \varphi[\overline{m}]\}$ is a subgroup of $\mathcal{M}^n$.*

(2) "*pp formulas with parameters are empty or define a coset of a definable subgroup*": *if $\bar{a} = a_j, \ldots, a_{n-1}$ is fixed in M^{n-j}, then $\varphi[M, \bar{a}] = \{\overline{m} \in M^j : \mathcal{M} \models \varphi[\overline{m}, \bar{a}]\}$ is either empty or a coset of the definable subgroup $\varphi[M, \overline{0}]$.*

(3) "*the sum of two pp-definable sets is pp-definable*": *given φ and ψ, we can find a pp formula χ such that $\chi[M, \bar{a}, \bar{b}] = \varphi[M, \bar{a}] + \psi[M, \bar{b}]$.*

(4) "*pp formulas factor across direct sums*": *if $\bar{a} \in M \oplus N$, (with a typical element $\langle a_j^0, a_j^1 \rangle$) then $\mathcal{M} \oplus \mathcal{N} \models \varphi[\bar{a}]$ if and only if $\mathcal{M} \models \varphi[\bar{a}^0]$ and $\mathcal{N} \models \varphi[\bar{a}^1]$.*

(5) "*Homomorphisms preserve pp formulas*": *if $\mathcal{M} \models \varphi[\bar{a}]$ then $\mathcal{N} \models \varphi[f(\bar{a})]$ (equivalently, $f[\varphi[M]] \subseteq \varphi[N]$).*

A homomorphism which reflects pp formulas is called *pure*, in which case it is an embedding. Clearly elementary embeddings are pure; and if $\mathcal{A}$ is a direct summand of $\mathcal{B}$, the natural inclusion of $\mathcal{A}$ in $\mathcal{B}$ is pure.

Theorem 1.4 (Baur [3], Monk [24], "pp elimination of quantifiers"). *Given a complete theory T of R-modules, every formula of ${}_R\mathcal{L}$ is equivalent modulo T to a Boolean combination of pp formulas.*

Thus, in any module $\mathcal{M}$, a definable subset of M^n (definable with parameters) is a Boolean combination of cosets of subgroups of M^n definable without parameters. It is this result that makes the model theory of modules so profitable algebraically: the definable sets are essentially objects of module theory; and it therefore turns out that a very wide range of model theoretic concepts have essentially algebraic formulations, and conversely. In fact considering the definable cosets only (rather than their complements as well) turns out to be sufficient for most model-theoretic purposes. For instance, elementary embeddings are clearly pure; and in many arguments it suffices to use pure embeddings instead of elementary embeddings. Similarly sufficiently saturated models are clearly pure-injective, and in many model-theoretic arguments pure-injective modules suffice.

In fact there is deeper theory underlying all these results. Each positive primitive formula of ${}_R\mathcal{L}$ defines a functor from the category of (left) R-modules to the category of abelian groups. It turns out that these functors are well behaved, and that many concepts of the model theory of modules have their most natural expression in the context of functor categories. In fact, a significant part of the development of the model theory of modules in recent years has gone in this direction and explicit references to model theory can disappear entirely. It would lead us too far astray to attempt to provide any significant details; instead we refer the reader to Prest [29]. Nonetheless, we are left with a strong motivation to understand the category of topological modules, and the functors on it.

There are some basic properties of direct sums and direct products which are important in the model theory of modules:

Theorem 1.5 (see, for instance, Prest [28, Section 2.5]**).** *Let $\mathcal{M}$ and $\mathcal{N}$ be R-modulestextup, and $\langle \mathcal{M}_i \rangle_{i\in I}$ a family of R-modules.*

(1) *If $\mathcal{M} \prec \mathcal{N}$ then $\mathcal{M}$ is elementarily equivalent to $\mathcal{M} \oplus (\mathcal{N}/\mathcal{M})$.*
(2) $\bigoplus_{i\in I} \mathcal{M}_i \equiv \prod_{i\in I} \mathcal{M}_i$
(3) *For any infinite I, $\mathcal{M}^{(I)} \equiv \mathcal{M}^{(\aleph_0)} \equiv \mathcal{M}^{\aleph_0} \equiv \mathcal{M}^I$*
(4) $\bigoplus_{i\in I} \mathcal{M}_i \preceq \prod_{i\in I} \mathcal{M}_i$

These results, together with the basic results of homological algebra relating the Hom functor with the direct sum and the direct product, begin to explain the power and significance of direct sums and direct products in the model theory of modules.

Definition 1.6. Let $\mathcal{N}$ be an R-module.

(1) $\mathcal{N}$ is *pure-injective* if it is injective over pure embeddings, that is, whenever $e\colon \mathcal{A} \to \mathcal{B}$ is a pure embedding of R-modules and $f\colon \mathcal{A} \to \mathcal{N}$ is a homomorphism, there is a homomorphism $\hat{f}\colon \mathcal{B} \to \mathcal{N}$ lifting f, $f = \hat{f} \circ e$.

(2) $\mathcal{N}$ is *equationally compact* if every system Σ of R-linear equations over N which is finitely solvable in $\mathcal{N}$ is actually solvable in $\mathcal{N}$.

Here Σ may consist of infinitely many equations, may involve infinitely many variables, and the equations may have nonzero constant terms from N. "Finitely solvable" means that every finite subset of Σ has a solution in $\mathcal{N}$.

(3) $\mathcal{N}$ is *pp-compact* if every set Φ of pp formulas over N which is finitely satisfiable in $\mathcal{N}$ is actually satisfiable in $\mathcal{N}$.

Here Φ may consist of infinitely many pp formulas, may involve infinitely many variables, and the formulas may have arbitrary parameters from N. "Finitely satisfiable" means that every finite subset of Φ is realized in $\mathcal{N}$.

Pure injectivity is a natural algebraic concept. Clearly injective modules are pure-injective, so $\mathbb{Q}$ and the Prüfer p-groups $\mathbb{Z}(p^\infty)$ are all pure-injective. Equational compactness and pp-compactness are natural model-theoretic concepts. Clearly $|R|^+$-saturated models are pp-compact; and pp compact structures are equationally compact. If $\mathcal{N}$ carries a compatible compact Hausdorff topology, then the solution sets of linear equations are closed sets, so such a module is clearly equationally compact. So for instance the circle group $\mathbb{R}/\mathbb{Z}$ and the group of p-adic numbers (p prime) are each equationally compact. Clearly any finite module is pp-compact.

Theorem 1.7. *Let $\mathcal{N}$ be an R-module. The following are equivalent*:

(1) $\mathcal{N}$ *is pure-injective*;
(2) $\mathcal{N}$ *is equationally compact*;
(3) $\mathcal{N}$ *is pp-compact.*

The equivalence between pure-injectivity and pp-compactness, coupled with the pp elimination of quantifiers, is what makes pure-injectivity a sufficient replacement for saturation in many arguments. Some of that flavour may be seen in the two arguments that we will outline below.

It seems clear that since the topological language involves references to open sets as well as to individuals that there will be no natural extension of the concept of equational compactness to the context of topological modules. But the other two concepts do hold some promise of generalization, and what we would like to see then is that the equivalence between them should still hold. For ordinary modules, the arguments in both directions are natural, and we provide brief sketches.

PP-COMPACTNESS IMPLIES PURE-INJECTIVITY. Let $e\colon \mathcal{A} \to \mathcal{B}$ be a pure embedding of R-modules and let $f\colon \mathcal{A} \to \mathcal{N}$ be a homomorphism into pp-compact $\mathcal{N}$. Establish a bijection $b \mapsto v_b$ between the elements of B and a set $\mathcal{V}$ of new, distinct variables. Let Φ be the "pp-type" of B over A in $\mathcal{B}$: the set of all pp formulas with parameters from A and free variables from $\mathcal{V}$ satisfied in $\mathcal{B}$ by assigning to each variable its corresponding element of B. Since e is pure, it is easy to see that Φ is finitely satisfiable in $\mathcal{A}$. Now replace each parameter $a \in A$ by $f(a) \in N$ to get a set of pp formulas $f(\Phi)$ with parameters in N. Since f is a homomorphism, it is easy to see that $f(\Phi)$ is finitely satisfiable in $\mathcal{N}$. Since $\mathcal{N}$ is pp-compact, $f(\Phi)$ is satisfied in $\mathcal{N}$, say by $\langle n_b \rangle_{b\in B}$. Define $\hat{f}\colon \mathcal{B} \to \mathcal{N}$ by sending each $b \in B$ to the corresponding element n_b of N. It is easy to check that $\hat{f}$ is a module homomorphism satisfying $f = \hat{f} \circ e$. □

PURE-INJECTIVITY IMPLIES PP-COMPACTNESS. Let Φ be a finitely satisfiable set of pp formulas over the pure-injective module $\mathcal{N}$. Then Φ is certainly consistent, so there is an elementary extension $\mathcal{N}'$ of $\mathcal{N}$ in which Φ has a solution. Elementary extensions are pure. Let $f\colon \mathcal{N} \to \mathcal{N}$ be the identity map. By pure-injectivity, f lifts to $\hat{f}\colon \mathcal{N}' \to \mathcal{N}$. The image of the solution of Φ in $\mathcal{N}'$ under $\hat{f}$ is clearly a solution of Φ in $\mathcal{N}$.

This "short proof" uses much too heavy machinery — full compactness to construct an elementary extension with a desired property. Prest [29, Theorem 4.1.4] takes a somewhat more elementary approach, showing that whenever Φ is a finitely satisfiable set of pp formulas over a module $\mathcal{M}$, the diagonal embedding of $\mathcal{M}$ into a suitably chosen reduced product $\mathcal{M}^I/\mathcal{F}$ is a pure embedding of $\mathcal{M}$ into a module in which Φ has a solution. □

It is these proofs that we would like to see generalize to the context of topological modules in the language $\mathcal{L}_{\mathrm{t}}$. A test of a possible definition of purity for topological modules would be whether it had associated concepts of pure-injectivity and pp-compactness that were related in a useful way.

2. Topological abelian groups and modules

A *topological abelian group* is a structure $\langle \mathcal{A}, \tau \rangle$ where $\mathcal{A} = \langle A, +, -, 0 \rangle$ is an abelian group and τ is a topology under which $+$ and $-$ are continuous. Virtually all of our interest is in Hausdorff topologies: for groups this is equivalent to $\{0\}$ being closed.

Convention D. All topological abelian groups and modules are assumed to be Hausdorff.

This is no great restriction, since "Hausdorff" can be axiomatized by a sentence of the language $\mathcal{L}_{\mathrm{t}}$ to be introduced later.

For each fixed ring R, a *topological R-module* is a structure $\langle \mathcal{M}, \sigma \rangle$ such that $\mathcal{M}$ is an R-module and τ is a topology on the underlying set M of $\mathcal{M}$ under which $+$, $-$ and each map $M \to M : m \mapsto r \centerdot m$ are all continuous. This is a restricted case of what is normally considered in the literature: in general, R should be a topological ring, and scalar multiplication should be a continuous map $R \times M \to M$. However the usual language for R-modules incorporates R into the language rather than in the structures themselves, so there is no way to express topological properties of R in the corresponding topological languages. Equivalently, we consider only topological modules over discrete topological rings.

For any two topological structures of the same type, a *topological embedding* is an embedding which is also a homeomorphism onto a subspace of the codomain. That is, the embedding must be continous and open onto its image. The condition "open onto its image" is important in and of itself; a continuous map between two spaces is called *proper* if it is open onto its image. One particular reason for the usefulness of this concept comes from the duality theory of locally compact abelian groups: the Pontryagin dual of an exact sequence of locally compact abelian groups need not be exact; but the dual of a proper exact sequence is proper exact (see, for instance, Armacost [1, [Proposition 9.14]].)

This raises the whole question of what the category **TopAb** of topological abelian groups or the category of topological modules might be, and what its properties are. The maps are just the continuous module homomorphisms. However, the resulting category is not an abelian category; and many associated concepts either fail to exist at all in general in **TopAb**, or are "unnatural" in one way or another. This gives us the first indication that the model theory of topological abelian groups will likely be very different from that of ordinary abelian groups and modules.

(1) Neither monomorphisms nor epimorphisms need be *normal* (that is, the kernel or cokernel of some map, respectively);
(2) Given a monomorphism $f \colon \langle \mathcal{A}, \sigma \rangle \to \langle \mathcal{B}, \tau \rangle$, it may not be possible to extend this to a short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$;
(3) There appears to be no natural definition of tensor product.

For instance, if σ' is a group topology on $\mathcal{A}$ strictly finer than σ, then the identity map $\mathrm{id} \colon \langle \mathcal{A}, \sigma' \rangle \to \langle \mathcal{A}, \sigma \rangle$ is a continuous homomorphism which is both mono and epi, but is neither normal mono nor normal epi; and this monomorphism cannot be extended to a short exact sequence. We might hope to improve the situation by insisting that all the maps be proper; but this is doomed to failure as the composition of proper maps need not be proper.

Nonetheless, there is some natural homological algebra in the categories of topological abelian groups or topological R-modules. Given two topological left R-modules $\langle \mathcal{A}, \sigma \rangle$ and $\langle \mathcal{B}, \tau \rangle$, we let

$$\mathrm{CHom}(\mathcal{A}, \mathcal{B}) = \{ f \in \mathrm{Hom}(\mathcal{A}, \mathcal{B}) : f \text{ is continuous} \}.$$

The pointwise sum of two continuous homomorphisms is continuous, so $\mathrm{CHom}(\mathcal{A}, \mathcal{B})$ is a subgroup of $\mathrm{Hom}(\mathcal{A}, \mathcal{B})$. The *compact-open topology* on this subgroup is the topology having as a sub-basis of open sets all sets of the form (K, U) where K is a compact subset of $\mathcal{M}$ and U is open in $\mathcal{N}$. Then $\mathrm{CHom}(\mathcal{A}, \mathcal{B})$ endowed with the compact-open toplology is a topological abelian group. There are three standard theorems relating $\mathrm{Hom}(-,-)$ to the direct sum and direct product of modules. Parallel, but not identical, results for $\mathrm{CHom}(-,-)$, the Tychonoff product of topological abelian groups, and the coproduct of topological abelian groups, can be found in exercises 2.10.3 and 2.10.4, in Dikranjan et al. [6, pp. 66 – 69]. These all generalize naturally to topological modules.

3. Weak structures for second-order logic

Henkin [11] introduced a method of approximating higher order logics in first-order logic, now usually called the *method of weak structures.* Associated with any first-order language $\mathcal{L}$ there is a second-order language, consisting of $\mathcal{L}$ augmented by a new, infinite set of variables (called *set variables*), two new quantifiers (universal and existential) that bind set variables, equality $=_2$ between set variables, and a new binary relation symbol $\in$ between individuals and sets. In the second-order logic of $\mathcal{L}$ the set quantifiers are intended to range over all subsets of the domain of an $\mathcal{L}$-structure. Syntactically the same is the two-sorted expansion of $\mathcal{L}$ called $\mathcal{L}_2$. A *weak structure* for the second-order logic of $\mathcal{L}$ is a two-sorted structure where the first sort is just an ordinary $\mathcal{L}$-structure and the second sort is some collection of subsets of the domain of the first sort. Now in two-sorted logic the set quantifiers are interpreted as ranging over the second sort, and $\in$ is interpreted as a relation between individuals of the first sort and individuals of the second sort. The logic of weak structures is just ordinary, two-sorted, first-order, logic.

Consider the sentence

$$\textbf{ext}: \qquad \forall X \forall Y \big(X =_2 Y \leftrightarrow \forall v (v \in X \leftrightarrow v \in Y) \big)$$

If the $\mathcal{L}_2$-structure $\langle \mathcal{M}, A \rangle$ is a model of **ext**, then we can replace each $a \in A$ by

$$\hat{a} = \{ m \in M : \langle \mathcal{M}, A \rangle \models m \in A \}$$

to get a weak structure for the second-order logic of $\mathcal{L}$ which is isomorphic as an $\mathcal{L}_2$-structure to $\langle \mathcal{M}, A \rangle$. The study of weak models of **ext** is the desired first-order approximation to the study of the full second-order logic of $\mathcal{L}$.

4. Languages for topological modules

The concept of "topology on a set X" is not even second-order over X ("the union of an arbitrary family of open sets is open"); however, the concept of "basis for a topology on a set X" is axiomatizable in weak structures:

$$\begin{aligned}\textbf{bas}: \quad & \forall v \exists V (v \in V) \\ & \wedge \forall v \forall U \forall V \big[(v \in U \wedge v \in V) \to \exists W (v \in W) \wedge \forall w \big(w \in W \to (w \in U \wedge w \in V) \big) \big]\end{aligned}$$

If $\langle \mathcal{M}, \sigma \rangle$ is a weak structure for $\mathcal{L}$, then it is a model of **bas** iff σ is a basis for a topology on the domain of $\mathcal{M}$. This is what suggested to the various authors mentioned in the introductory paragraphs that one should seek out the formulas φ of $\mathcal{L}_2$ that are *invariant with respect to basis*: formulas such that whenever σ and σ' are two bases for the same topology on the domain of $\mathcal{M}$,

$$\langle \mathcal{M}, \sigma \rangle \models \varphi \iff \langle \mathcal{M}, \sigma' \rangle \models \varphi.$$

These are precisely the formulas of $\mathcal{L}_\mathrm{t}$.

Definition 4.1. Let φ be a formula of $\mathcal{L}_2$ written in *negation normal form*, that is, it is written using only the propositional connectives $\neg, \wedge$, and $\vee$, and each occurrence of $\neg$ is attached to an atomic formula. φ is *positive* in U if each occurrence of U is in a negation-free atomic formula $\mathrm{t} \in U$, and φ is *negative* in V if each occurrence of V is in a negated atomic formula $\neg(\mathrm{t} \in V)$.

Definition 4.2. The set of all $\mathcal{L}_\mathrm{t}$ formulas is the smallest set $\mathcal{W}$ of $\mathcal{L}_2$ formulas containing the atomic $\mathcal{L}$ formulas and the atomic $\mathcal{L}_2$-formulas of the form $\mathbf{t} \in V$, closed under $\neg, \wedge$, and existential quantifiers of the first sort; such that if φ is in $\mathcal{W}$, $\mathbf{t}$ is an $\mathcal{L}$-term, V is a variable of the second sort, and φ is positive in V, then $\forall V(\mathbf{t} \in V \to \varphi)$ is in $\mathcal{W}$; and such that if φ is in $\mathcal{W}$, $\mathbf{t}$ is an $\mathcal{L}$-term, V is a variable of the second sort, and φ is negative in V, then $\exists V(\mathbf{t} \in V \wedge \varphi)$ is in $\mathcal{W}$.

Note that the atomic $\mathcal{L}_2$-formulas of the form $V_i = V_j$ *are not* formulas of $\mathcal{L}_\mathrm{t}$.

We abbreviate $\forall V(\mathbf{t} \in V \to \varphi)$ as $(\forall V)_\mathbf{t}\varphi$ and $\exists V(\mathbf{t} \in V \wedge \varphi)$ as $(\exists V)_\mathbf{t}\varphi$, intended to be read as "for every open neighborhood V of $\mathbf{t}$, φ holds" and "for some open neighborhood V of $\mathbf{t}$, φ holds" respectively.

Theorem 4.3.
(1) (McKee [23]) *$\mathcal{L}_\mathrm{t}$ formulas are invariant with respect to basis*;
(2) (McKee [21, 23], Garavaglia [10]) *An $\mathcal{L}_2$ sentence is invariant with respect to basis iff it is $\mathcal{L}_2$-equivalent to an $\mathcal{L}_\mathrm{t}$ sentence in topological structures.*

Since $\mathcal{L}_2$ is an ordinary first-order logic, we immediately obtain the compactness theorem and the Löwenheim–Skolem theorem for $\mathcal{L}_\mathrm{t}$ ([10, 31]); and Ziegler showed that in fact $\mathcal{L}_\mathrm{t}$ satisfies a Lindström's theorem for topological structures.

Garavaglia [10] and Flum and Ziegler [9] enumerate many properties than can be expressed in $\mathcal{L}_\mathrm{t}$, amongst which: continuity and openness of functions (but not closedness), openness or closedness of definable sets, the closure of a definable set, separation axioms T_0 through $T_{2\frac{1}{2}}$ and regularity, so T_3 (but no higher), and discrete or trivial topologies.

It follows from the Löwenheim–Skolem theorem for $\mathcal{L}_\mathrm{t}$ that properties such as "normal," "connected," "compact," or "locally compact" are not expressible in $\mathcal{L}_\mathrm{t}$, even by a set of sentences.

The topology on a topological abelian group A or module is special: for each $a \in A$ the translation $x \mapsto a + x$ is a homeomorphism and so the topology is completely determined by the set of (open) neighborhoods of 0: $U \subseteq A$ is open iff for every $a \in U$ there is a neighborhood N of 0 such that $a + N \subseteq U$. Topologies of this sort can be described in terms of monotone systems:

Definition 4.4. A (k-)*monotone system* on a set A is a collection ν of nonempty subsets of A (of A^k) such that $U \in \nu$ and $U \subseteq V \subseteq A$ ($U \subseteq V \subseteq A^k$) implies that $V \in \nu$.

If μ is any collection of nonempty subsets of A then $\hat{\mu}$ denotes the smallest monotone system on A containing μ.

The set of all neighborhoods of 0 in a topological module forms a 1-monotone system.

Flum and Ziegler [9, [pp. 52 et seq.] develop the basic theory of a logic $\mathcal{L}_{\mathrm{m}}$ for monotone systems completely parallel to the development of $\mathcal{L}_{\mathrm{t}}$; in particular $\mathcal{L}_{\mathrm{m}}$ satisfies the compactness theorem and Löwenheim−Skolem theorem, and a Lindström's Theorem.

The language $\mathcal{L}_{\mathrm{m}}$ is defined just as the language $\mathcal{L}_{\mathrm{t}}$, except that the set quantifiers now just appear as $\forall V$ or as $\exists V$, instead of as $(\forall V)_t$ or as $(\exists V)_t$. The appropriate analogue of Theorem 4.3 holds, cf. Flum and Ziegler [9, Part I: 8.5, 8.6, 8.7].

The concept of "topological abelian group" ("topological left R-module") is easy to axiomatize in the languages $\mathcal{L}_{\mathrm{t}}$ and $\mathcal{L}_{\mathrm{m}}$. In addition, for any formula $\varphi(\bar{u}, \overline{V})$ of $\mathcal{L}_{\mathrm{t}}$ one can construct a formula $\varphi'(\bar{u}, \overline{V})$ of $\mathcal{L}_{\mathrm{m}}$ such that for all topological modules $\langle \mathcal{A}, \sigma \rangle$ and corresponding neighborhood basis μ at 0, all $\bar{a}$ in A and all open neighborhoods $\overline{\mathrm{U}}$ of 0 in $\langle \mathcal{A}, \sigma \rangle$,

$$\langle \mathcal{A}, \sigma \rangle \models \varphi[\bar{a}, \overline{\mathrm{U}}] \quad \Leftrightarrow \quad \langle \mathcal{A}, \mu \rangle \models \varphi'[\bar{a}, \overline{\mathrm{U}}],$$

and conversely. As a result, all the basic definitions and theorems for the model theory of topological modules in $\mathcal{L}_{\mathrm{t}}$ carry over, *mutatis mutandis*, to the model theory of topological modules in $\mathcal{L}_{\mathrm{m}}$, and we will not state them explicitly.

We say that two $\mathcal{L}_{\mathrm{t}}$ structures are $\mathcal{L}_{\mathrm{t}}$-elementary equivalent ("$\equiv_{\mathrm{t}}$") if they satisfy exactly the same $\mathcal{L}_{\mathrm{t}}$ sentences; and we say that two $\mathcal{L}_{\mathrm{m}}$ structures are $\mathcal{L}_{\mathrm{m}}$-elementary equivalent ("$\equiv_{\mathrm{m}}$") if they satisfy exactly the same $\mathcal{L}_{\mathrm{m}}$ sentences. Clearly this is a weaker concept than elementary equivalence as two-sorted structures.

Just as for the model theory of ordinary modules, there is a partial elimination of quantifiers for theories of topological modules.

Definition 4.5 (Garavaglia [10]**).** Let $\mathcal{L} = {}_R\mathcal{L}$, the usual language for left R-modules.

A *topological positive primitive formula* (tppf) is a formula of $\mathcal{L}_{\mathrm{t}}$ of the form

$$Q_1 Q_2 \dots Q_n \varphi$$

where φ is a conjunction of atomic $\mathcal{L}_{\mathrm{t}}$ formulas (hence, without loss of generality, of the form $\mathbf{t} = 0$ or of the form $\mathbf{t} \in V$ for some term $\mathbf{t}$ of $\mathcal{L}$), and each Q_i is a quantifier $\exists u$ or a quantifier $(\forall V)_0$ for some variables u, V of the first and second sort respectively.

Clearly the translation between $\mathcal{L}_{\mathrm{t}}$ and $\mathcal{L}_{\mathrm{m}}$ described previously takes topological positive primitive formulas to monotone positive primitive formulas, and conversely.

Theorem 4.6 (Garavaglia [10], **Kucera** [17, Corollary 3.7]**).** *Let $\langle \mathcal{M}, \sigma \rangle$ be a topological left R-module.*

Let $\varphi(\overline{x})$ be a formula of $\mathcal{L}_{\mathrm{t}}$ with only individual variables free. Then there is $\psi(\overline{x})$ with the same free variables, ψ a Boolean combination of topological positive primitive formulas, such that for all $\overline{m} \in M$, $\langle \mathcal{M}, \sigma \rangle \models \varphi[\overline{m}]$ if and only if $\langle \mathcal{M}, \sigma \rangle \models \psi[\overline{m}]$.

It is this theorem that gives us some hope that there might be a reasonable and useful "model theory of topological modules." We need to wait until we have some more material on saturated modules before we can show (Corollary 5.3) that tppfs define subgroups, as they do in ordinary theories of modules. Although historically it has been the language $\mathcal{L}_{\mathrm{t}}$ which has received the most attention, the language $\mathcal{L}_{\mathrm{m}}$ is often more convenient (as Flum and Ziegler noted); and we will see in the next sections at least one reason for preferring it.

5. Model theory of topological modules in two sorted logic

The general philosophy behind the "weak structures" approach to second-order logic, coupled with the characterization of invariant formulas, suggests that we should ascribe to a topological module $\langle \mathcal{M}, \tau \rangle$ some model theoretic property whenever $\langle \mathcal{M}, \beta \rangle$ has that property in two-sorted first-order logic, for some basis β for τ. Similarly, if ν is the set of all neighborhoods of 0, we ascribe a property to the monotone structure $\langle \mathcal{M}, \nu \rangle$ if $\langle \mathcal{M}, \mu \rangle$ has that property in two-sorted first order logic for some basis μ for ν.

In particular, $\langle \mathcal{M}, \sigma \rangle$ has cardinality κ means that M has cardinality κ and σ has a basis of cardinality κ. We say that $\langle \mathcal{M}, \sigma \rangle$ is (κ-)saturated if and only if $\langle \mathcal{M}, \sigma' \rangle$ is (κ-)saturated as a two-sorted structure for some basis σ' of σ. Many of the key results of [10] and [17] were obtained by using structures that were saturated [or special] in this way. Note that while saturation is a property of the topological structure, it is not a property that is invariant with respect to basis. Clearly if $\langle \mathcal{M}, \sigma \rangle$ is κ-saturated as a two-sorted structure then $\mathcal{M}$ is saturated as an $\mathcal{L}$-structure.

Proposition 5.1 (Flum and Ziegler [9, Part I: Lemma 4.7]**).** *Let $\langle \mathcal{M}, \sigma \rangle \equiv_{\mathrm{t}} \langle \mathcal{N}, \tau \rangle$ be saturated structures of the same cardinality. Then they are topologically isomorphic.*

It is worthwhile to re-iterate exactly what the hypotheses mean: As topological structures, $\langle \mathcal{M}, \sigma \rangle$ and $\langle \mathcal{N}, \tau \rangle$ satisfy the same $\mathcal{L}_{\mathrm{t}}$ sentences. We can find bases σ' and τ' for σ and τ respectively, such that $|M| = |\sigma'| = |N| = |\tau'| = \kappa$ and $\langle \mathcal{M}, \sigma' \rangle$ and $\langle \mathcal{N}, \tau' \rangle$ are κ-saturated as two-sorted structures.

Flum and Ziegler prove this result only for $\kappa = \aleph_0$, but a straightforward modification of their back-and-forth argument proves the same result for saturated structures of any infinite cardinality. In particular, in constructing the back-and-forth system, one must consider partial homeomorphisms of cardinality less than κ, and then use the κ-saturation to show that the resulting system is closed under unions of chains of cardinality less than κ.

Clearly if $\langle \mathcal{M}, \sigma \rangle$ is a topological module, σ is some basis for τ, and $\langle \mathcal{M}, \sigma \rangle \equiv_2 \langle \mathcal{M}', \sigma' \rangle$ or $\langle \mathcal{M}, \sigma \rangle \preceq_2 \langle \mathcal{M}', \sigma' \rangle$ or $\langle \mathcal{M}', \sigma' \rangle \preceq_2 \langle \mathcal{M}, \sigma \rangle$ as two sorted structures, then $\langle \mathcal{M}, \sigma \rangle$ and $\langle \mathcal{M}', \sigma' \rangle$ are $\mathcal{L}_{\mathrm{t}}$-elementarily equivalent. This leads to some interesting constructions; in particular see Example 5.7.

Note that in spite of the general principle enunciated above, it would *not* be useful to try to define "elementary substructure" by saying that "$\langle \mathcal{M}, \sigma \rangle$ is an elementary substructure of $\langle \mathcal{N}, \tau \rangle$ iff for some basis α of σ and some basis β of τ, $\langle \mathcal{M}, \alpha \rangle \preceq_2 \langle \mathcal{N}, \beta \rangle$," for then the resulting relation might not even be transitive.

A standard simplifying assumption of ordinary one-sorted model theory is that an embedding can always be replaced by an inclusion of structures. Of course we can make the same replacement for a two-sorted embedding; but not for embeddings between weak structures. The individuals of the second sort of a weak structure are constrained to be subsets of the first sort. The best that we can do is this: if $f\colon \langle \mathcal{M}, \alpha\rangle \to \langle \mathcal{N}, \beta\rangle$ is an embedding of weak structures, then we may assume without loss of generality that on the first sort, f is an inclusion of structures $\mathcal{M} \subseteq \mathcal{N}$. On the second sort, f is a one-to-one map identifying certain subsets of M (the elements of α) with certain subsets of N (which are elements of β). The fact that f respects atomic formulas implies that for each $A \in \alpha$, $f(A) \cap M = A$. But note that in general, $f(A) \supsetneq f[A] = \{f(m) : m \in A\}$. So in particular if these are topological structures, then α will be contained in the subspace topology induced from β, but this subspace topology might be much finer than α. An important example of this is seen in Corollary 5.4. Further developments of these ideas are seen in the discussion of continuous homomorphisms in Proposition 6.1 and following.

It helps to consider a concrete example, for instance the natural inclusion of the rational group as a subspace of the real group, both with the Euclidean topology. If we wish to consider this embedding as a two-sorted embedding, we have to map open sets of $\mathbb{Q}$ to open sets of $\mathbb{R}$. There is no canonical way of doing this for all open sets of $\mathbb{Q}$. It is however possible to do so for particular choices of basis: we could take instead the basis for the Euclidean topology on $\mathbb{Q}$ consisting of all bounded intervals with rational endpoints, and match each such interval with the corresponding interval in $\mathbb{R}$.

Theorem 5.2. (1) (Flum and Ziegler [9, Part II, Lemma 1.18]) *Let $\langle \mathcal{M}, \sigma\rangle$ be a topological module with a basis μ for the neighborhoods at* 0 *such that $\langle \mathcal{M}, \mu\rangle$ is $\aleph_1$-saturated. Then τ is closed under countable intersections.*

(2) (Garavaglia [10]) *Let $\langle \mathcal{M}, \sigma\rangle$ be a topological module with a basis μ for the neighborhoods at* 0 *such that $\langle \mathcal{M}, \mu\rangle$ is $\aleph_1$-saturated Then $\hat{\mu}$ has a basis at* 0 *consisting of subgroups.*

The following is folklore, but has not been stated before with a clear proof.

Corollary 5.3 (Enns [8, Corollary 4.29]**).** *If $\varphi(\bar{v})$ is a topological positive primitive formula with no set variables free, then φ defines a subgroup of $\mathcal{M}^n$ (for some n) for any topological module $\langle \mathcal{M}, \sigma\rangle$. If $\varphi(\bar{v}, \bar{a})$ is a is a tppf with no set variables free and parameters from M, then it is either unsatisfiable in $\langle \mathcal{M}, \sigma\rangle$, or defines a coset of the subgroup defined by $\varphi(\bar{v}, \overline{0})$.*

Proof. Let $\langle \mathcal{N}, \tau\rangle$ be an $\aleph_1$-saturated two-sorted elementary extension of $\langle \mathcal{M}, \sigma\rangle$. Then τ has a basis τ_0 at 0 consisting of subgroups of $\mathcal{M}$. Consider the structure $\langle \mathcal{N}, \tau_0\rangle$. Atomic formulas $\mathbf{t} = 0$ and $\mathbf{t} \in A$ (for any set parameter $A \in \tau_0$) clearly define subgroups since terms are linear. Then the conjunction of atomic formulas (with set parameters) defines an intersection of subgroups; individual existential quantification defines a projection of a subgroup; and universal set quantification defines an intersection of a family of subgroups. Hence any tppf even with set parameters defines a subgroup of $\mathcal{N}^n$ in the structure $\langle \mathcal{N}, \tau_0\rangle$. In particular, tppfs without set parameters define subgroups. But to say that a definable set is a subgroup is itself an $\mathcal{L}_{\mathrm{t}}$-formula, so by the invariance of $\mathcal{L}_{\mathrm{t}}$ formulas with respect to basis, they also define subgroups in $\langle \mathcal{N}, \tau\rangle$ and hence in $\langle \mathcal{M}, \sigma\rangle$. □

We cannot conclude anything in general about the topological properties of definable subgroups. Noting that open subgroups are closed, we see that although atomic formulas (even with open subgroup parameters) define closed subgroups, and both conjunctions and universal set quantifiers ($(\forall V)_0$) define intersections of closed subgroups, hence closed; it is not true in general that a projection ($(\exists w)$) of a closed subgroup is closed. For instance, consider $\mathbb{Z}$ with the p-adic topology, p a prime. For primes $q \neq p$, $q\mathbb{Z}$ is definable (by $\exists w(qw = v)$), and is not closed; in fact it is a dense subgroup.

Corollary 5.4 (Enns [8, Corollary 2.10]**).** *Let $\langle \mathcal{M}, \sigma \rangle$ be a topological module with a basis μ for the neighborhoods at 0 such that $\langle \mathcal{M}, \mu \rangle$ is $\aleph_1$-saturated. Let $\mathcal{M}_0$ be a countable submodule of $\mathcal{M}$. Then the subspace topology on M_0 is the discrete topology.*

Proof. For each $0 \neq b \in M_0$, $U_b = M_0 \setminus \{b\}$ is an open neighborhood of 0 (in M_0) since τ is Hausdorff. Hence $\{0\} = \bigcap_{0 \neq b \in M_0} U_b$ is open since M_0 is countable. Thus the subspace topology on M_0 is discrete. □

Note that this argument applies to any kind of $\aleph_1$-saturated topological structure, not just to topological modules. So in general we are warned away from using two-sorted embeddings for a study of topological structures, as they will not necessarily be embeddings onto subspaces.

Definition 5.5 (Flum and Ziegler [9, Part II, 2.3]**).** A topological abelian group is *locally pure* if for each $n > 0$ it satisfies the following $\mathcal{L}_{\mathrm{m}}$ sentence:

$$\forall U \exists V \forall u \exists v \, [n \cdot u \in V \rightarrow (v \in U \wedge n \cdot u = n \cdot v)]$$

Abelian groups with the discrete or the trivial topology are vacuously locally pure; groups with a topology induced by a linear order are locally pure; and the group of integers with any n-adic topology ($n \geq 2$) is locally pure.

Lemma 5.6 (Flum and Ziegler [9, Part II, 2.4]**).** *An $\aleph_1$-saturated locally pure topological abelian group has a basis at 0 consisting of pure subgroups.*

Example 5.7. Consider the rational group $\mathbb{Q}$ with the usual Euclidean topology $\mathcal{E}$. Flum and Ziegler [9, Part II, 2.9] give a complete axiomatization for the $\mathcal{L}_{\mathrm{m}}$ theory of $\langle \mathbb{Q}, \mathcal{E}_0 \rangle$ (where $\mathcal{E}_0$ is the set of all Euclidean neighborhoods of 0 in $\mathbb{Q}$):

"torsion-free, divisible, nonzero, Hausdorff, nondiscrete, and locally pure."

(Note that Flum and Ziegler fail to mention "nondiscrete.")

It follows immediately that the group of the reals with the Euclidean topology is $\mathcal{L}_{\mathrm{m}}$-equivalent to $\langle \mathbb{Q}, \mathcal{E}_0 \rangle$; and similarly for any $\mathbb{Q}^n$ with the Euclidean topology.

Now the model theory of the group $\mathbb{Q}$ is very well understood, and quite simple. The groups elementarily equivalent to $\mathbb{Q}$ are precisely the nonzero $\mathbb{Q}$-vector spaces; so the theory is $\aleph_1$-categorical. A model is κ-saturated iff its dimension as a vector space is $\geq \kappa$.

Consider some two-sorted elementary extension $\langle \mathcal{M}, \mu \rangle$ of $\langle \mathbb{Q}, \mathcal{E}_0 \rangle$ which is $\aleph_1$-saturated. Then the monotone system generated by μ has a basis μ' of open, pure, subgroups by Lemma 5.6. As well as **bas**, $\langle \mathcal{M}, \mu' \rangle$ satisfies sentences of two-sorted logic asserting that each $U \in \mu$ is an open, pure, subgroup. By the downward Löwenheim – Skolem theorem for two-sorted logic, $\langle \mathcal{M}, \mu' \rangle$ has a countable elementary submodule $\langle \mathcal{M}_0, \mu_0 \rangle$, that is, the underlying set of $\mathcal{M}_0$ is countable and μ_0 is countable. This two-sorted structure is isomorphic to a weak structure $\langle \mathcal{M}_0, \mu_0' \rangle$. It follows that $\langle \mathcal{M}_0, \mu_0' \rangle$ is a countable topological abelian group $\mathcal{L}_{\mathrm{m}}$-equivalent to the rational group, and with a neighborhood basis at 0 consisting of open pure subgroups. Now a pure subgroup of a $\mathbb{Q}$-vector space is just a vector subspace; and a Hausdorff topology on a finite dimensional vector space generated by subspaces will be discrete. So $\mathcal{M}_0 \cong \mathbb{Q}^{(\aleph_0)}$, and μ_0 contains no finite dimensional subspaces. There are many possibilities for such, amongst which a natural example is the set μ_{cof} of all subspaces of finite codimension. This gives a natural countable model of the $\mathcal{L}_{\mathrm{m}}$ theory of $\langle \mathbb{Q}, \mathcal{E}_0 \rangle$ which looks quite different from the Euclidean topology on $\mathbb{Q}$ or $\mathbb{R}$, and which does not have $\langle \mathbb{Q}, \mathcal{E}_0 \rangle$ as a subspace. Although we have retained $\mathcal{L}_{\mathrm{m}}$-elementary equivalence at all stages, none of the two-sorted constructions preserve topologies. In fact, any finite dimensional subspace is discrete. There is a wide variety of similar models. Let $\mathcal{F}$ be any filter on the collection of all vector subspaces of $\mathbb{Q}^{(\omega)}$ containing μ_{cof}. Then $\mathcal{F}$ generates a group topology on $\mathbb{Q}^{(\omega)}$ that is $\mathcal{L}_{\mathrm{m}}$ equivalent to $\langle \mathbb{Q}, \mathcal{E}_0 \rangle$.

We make no claim as to what the κ-saturated elementary extensions of $\langle \mathbb{Q}, \mathcal{E}_0 \rangle$ for infinite κ might be; but we do note that $\mathbb{Q}^{(\kappa)}$ with the topology generated by subspaces of codimension less than κ is $\mathcal{L}_{\mathrm{m}}$-elementarily equivalent to $\langle \mathbb{Q}, \mathcal{E}_0 \rangle$, with a basis at 0 consisting of pure subgroups which is closed under intersections of families of size less than κ.

6. Embeddings and purity relative to $\mathcal{L}_{\mathrm{t}}$ or $\mathcal{L}_{\mathrm{m}}$

In the preceding section we saw some use for two-sorted maps between topological structures. This is unnatural from a topological standpoint. Indeed, the continuity of a map between topological structures is not defined in terms of taking open sets forward, but rather in terms of bringing open sets back. It is in this context of continuous homomorphisms that we would hope to find natural definitions of pure embedding.

There is a natural analogue to the homomorphism property for ordinary pp formulas (cf. Prest [28, Lemma 2.7]).

Proposition 6.1 (Enns [8, Lemma 4.24]**).** *Let $\langle \mathcal{M}, \sigma \rangle$ and $\langle \mathcal{N}, \tau \rangle$ be topological modules and let $f\colon M \to N$ be a continuous homomorphism. Then for every topological positive primitive formula $\varphi(\bar{v}, \overline{V})$, every $\overline{m} \in M$, and every $\overline{U} \in \tau$,*

$$\langle \mathcal{M}, \sigma \rangle \models \varphi\big[\overline{m}, f^{-1}[\overline{U}]\big] \implies \langle \mathcal{N}, \tau \rangle \models \varphi\left[f(\overline{m}), \overline{U}\right].$$

Proof. This follows by a simple induction on the length of the quantifier prefix of the topological positive primitive formula φ. □

Note the peculiar role played by the variables of the second sort: they are not "carried upwards," but rather downwards. As a special case, if $\varphi(\bar{v})$ is a tppf with no set variables free and f is a continuous homomorphism, then $\langle \mathcal{M}, \sigma \rangle \models \varphi[\overline{m}] \implies \langle \mathcal{N}, \tau \rangle \models \varphi[f(\overline{m})]$. We do not know how to characterize those R-module homomorphisms which always preserve tpp formulas with no set variables free. For later reference, we will call these $\mathcal{L}_{\mathrm{t}}$-*homomorphisms*.

Corollary 6.2. *Let $\langle \mathcal{M}, \sigma \rangle$ and $\langle \mathcal{N}, \tau \rangle$ be topological modules with $\langle \mathcal{M}, \sigma \rangle$ a submodule and a subspace of $\langle \mathcal{N}, \tau \rangle$. Then for all $\overline{m} \in M$ and open sets $\overline{U} \in \tau$.*

$$\langle \mathcal{M}, \sigma \rangle \models \varphi[\overline{m}, \overline{U} \cap M] \implies \langle \mathcal{N}, \tau \rangle \models \varphi[f(\overline{m}), \overline{U}].$$

The fact that homomorphisms preserve pp formulas upwards is one motivation for the definition of purity: a pure embedding is one that reflects pp formulas downwards. However there is an immediate problem to making the obvious generalization to topological modules. Consider the case of the very simple topological positive primitive formula $\psi(V) = \exists u\,(u \in V)$ with only the set variable V free.

Proposition 6.3 (Enns [8, Theorem 4.35]**).** *Let $\langle \mathcal{M}, \sigma\rangle$ and $\langle \mathcal{N}, \tau\rangle$ be topological modules and $f\colon M \to N$ a continuous homomorphism. If for every $U \in \tau$,*

$$\langle \mathcal{N}, \tau\rangle \models (\exists u\,(u \in V))[U] \implies \langle \mathcal{M}, \sigma\rangle \models (\exists u\,(u \in V))\big[f^{-1}[U]\big],$$

then $f[M]$ is dense in N.

Proof. Clearly $\langle \mathcal{N}, \tau\rangle \models (\exists u\,(u \in V))[U]$ just asserts that U is a nonempty open subset of N; and then the assumption implies that $f^{-1}[U]$ is a nonempty open subset of M. But $f^{-1}[U]$ is nonempty iff $f[M] \cap U$ is nonempty. Hence $f[M]$ is dense in N. $\square$

Clearly if we wish to have a useful model theory of topological modules, this is much too strong a restriction to impose. But there is an easy way out of this particular problem; one that has been alluded to before. If we use the monotone language rather than the topological language, then the above proof disappears, as the formula $\exists u\,(u \in V)$ is trivially true of any neighborhood of 0.

However a slightly more complicated formula points out some serious problems with carrying through set parameters even in $\mathcal{L}_{\mathrm{m}}$-formulas. If X is a set definable in a topological module, say by a formula $\psi(v)$, then the closure of X is also definable, by the $\mathcal{L}_{\mathrm{m}}$-formula $\mathbf{cl}\,\psi(v) = \forall W \exists w\big(w - v \in W \wedge \psi(w)\big)$. In particular, if ψ is a monotone positive primitive formula, so is $\mathbf{cl}\,\psi(v)$. If we allow set parameters we encounter the following problem. Consider the monotone positive primitive formula

$$\psi(v, V) = \forall W \exists w(w \in W \wedge w - v \in V),$$

that is, "$v \in \mathbf{cl}\,V$."

Proposition 6.4. *Let $\langle \mathcal{M}, \mu\rangle$ and $\langle \mathcal{N}, \nu\rangle$ be topological modules* (*considered as monotone structures*) *and $f\colon M \to N$ a nonzero, continuous homomorphism. Then f reflects all instances of "$v \in \mathbf{cl}\,V$" if and only if f is an open map.*

Proof. Suppose that W is a nonempty open set of the topology generated by μ, and that $f[W]$ is *not* open in $\langle \mathcal{N}, \nu\rangle$, that is, that $N \setminus f[W]$ is *not* closed. So we can chose an element m of $\mathbf{cl}_N(N \setminus f[W]) \cap f[W] \neq \varnothing$. Now for any $k \in f[M]$, $n \mapsto k + n$ is a homeomorphism of N onto itself sending $f[M]$ to $f[M]$ and $N \setminus f[M]$ to $N \setminus f[M]$. So we can assume without loss of generality by a suitable translation of W that $f(m) \neq 0$. Since Hausdorff groups are regular, we can choose a *closed* neighborhood $A \in \nu$ of 0 such that $f(m) \notin A$. Set $U = A \cup (N \setminus f[W])$, a neighborhood of 0.

Then trivially $f(m) \in \mathbf{cl}_N(U)$. Suppose that $m \in \mathbf{cl}_M(f^{-1}[U])$. We have

$$\begin{aligned}\mathbf{cl}_M(f^{-1}[U]) &= \mathbf{cl}_M(f^{-1}[A \cup (N \setminus f[W])]) \\ &\subseteq \mathbf{cl}_M(f^{-1}[A]) \cup \mathbf{cl}_M(M \setminus W) = \mathbf{cl}_M(f^{-1}[A]) \cup (M \setminus W)\end{aligned}$$

(the latter since $M \setminus W$ is closed). Thus $m \in \mathbf{cl}_M(f^{-1}[A])$ since $m \in W$ and so $f(m) \in \mathbf{cl}_N(A) = A$ by the continuity of f, contrary to the choice of A.

Conversely suppose that f is an open map. We want to see that for all $U \in \nu$ and all $m \in M$, $f(m) \in \mathbf{cl}_N(U) \Rightarrow m \in \mathbf{cl}_M(f^{-1}[U])$. Since in particular $f[M]$

is an open submodule, it is clopen. Hence for any $A \subseteq N$, $\mathbf{cl}_N(A) \cap f[M] = \mathbf{cl}_N(A \cap f[M])$. So $f(m) \in \mathbf{cl}_N(U)$ implies $f(m) \in \mathbf{cl}_N(U \cap f[M])$. Let W be open in the topology of $\langle \mathcal{M}, \mu \rangle$ with $m \in W$, so that $f[W]$ is open in the topology of $\langle \mathcal{N}, \nu \rangle$. Hence, since $f(m) \in f[W]$, $f[W] \cap (U \cap f[M]) \neq \varnothing$. Thus $W \cap f^{-1}[U] \neq \varnothing$, and so $m \in \mathbf{cl}_M(f^{-1}[U])$. □

So in spite of the promise of Lemma 6.1, we must abandon the idea of considering formulas with free set variables in any definition of topological purity.

Therefore the only decision remaining is what sort of topological constraints to put on the inclusion of $\langle \mathcal{M}, \mu \rangle$ as a submodule of $\langle \mathcal{N}, \nu \rangle$: either there is no relation between μ and ν; or $\mu \supseteq \nu \restriction M$; or $\mu = \nu \restriction M$. Now based on the types of arguments developed by the second author in [17] (e.g., the development of the basics of stability theory) it appears that there might be some sort of interesting "pure" model theory that can be developed under the weakest of these alternatives. In the same paper the second author proposed a definition of $\mathcal{L}_\mathrm{t}$-elementary embedding that placed no topological constraints on the maps, and only required that formulas with only individual variables free be preserved and reflected. At about the same time, Majewski [20] put forward a similar definition for elementary embedding, but requiring that the maps in question be topological embeddings. Neither author offered a single example. It is hard to see how such a development would contain much of topological interest. It appears that the most natural definition would be to take the third alternative, namely that $\langle \mathcal{M}, \mu \rangle$ should be a subspace of $\langle \mathcal{N}, \nu \rangle$, but we do explore two possible definitions.

Definition 6.5. Let $\langle \mathcal{M}, \sigma \rangle$ and $\langle \mathcal{N}, \tau \rangle$ be topological modules with $\mathcal{M}$ a submodule of $\mathcal{N}$.

(1) $\langle \mathcal{M}, \sigma \rangle$ is *weakly $\mathcal{L}_\mathrm{t}$-pure* in $\langle \mathcal{N}, \tau \rangle$ if for all tppfs $\varphi(\bar{v})$ with no set variables free, and all $\overline{m} \in M$,
$$\langle \mathcal{M}, \sigma \rangle \models \varphi[\overline{m}] \iff \langle \mathcal{N}, \tau \rangle \models \varphi[\overline{m}].$$
(2) $\langle \mathcal{M}, \sigma \rangle$ is *$\mathcal{L}_\mathrm{t}$-pure* in $\langle \mathcal{N}, \tau \rangle$ if it is weakly $\mathcal{L}_\mathrm{t}$-pure and in addition, σ is the subspace topology induced by τ.

An intermediate case would only require that $\sigma \supseteq \tau \restriction M$.

We choose *not* to call this a "topological(ly) pure submodule." Several people, most notably Peter Loth, have been exploring various concepts of purity in the context of locally compact abelian groups. We will discuss this further in Section 9.

An immediate problem for any of the potential definitions of pure-embedding for topological modules considered here is that of *existence*: does an arbitrary topological module have a proper pure extension? does it have a pure extension satisfying some reasonable conditions? What we have in mind here is the problem of finding some substitute for the use of elementary extensions or the use of reduced products in the proof of "pure-injectivity implies pp-compactness," Theorem 1.7. We have some limited results, the two following and Corollary 7.6.

Proposition 6.6. *Let $\langle \mathcal{M}, \sigma \rangle$ be a topological module and α a basis for σ. Suppose that the weak structure $\langle \mathcal{N}, \beta \rangle$ is a two-sorted elementary extension of $\langle \mathcal{M}, \alpha \rangle$, and let τ be the topology generated by β as a basis. Then $\langle \mathcal{N}, \tau \rangle$ is a weak $\mathcal{L}_\mathrm{t}$-pure extension of $\langle \mathcal{M}, \sigma \rangle$.*

PROOF. For $\overline{m} \in M$ and tpp formula φ,

$$\begin{aligned}
\langle \mathcal{N}, \tau \rangle \models_t \varphi[\overline{m}] &\Leftrightarrow \langle \mathcal{N}, \beta \rangle \models_2 \varphi[\overline{m}] &&\text{(invariance with respect to basis)}\\
&\Leftrightarrow \langle \mathcal{M}, \alpha \rangle \models_2 \varphi[\overline{m}] &&\text{(two-sorted elementary substructure)}\\
&\Leftrightarrow \langle \mathcal{M}, \sigma \rangle \models_t \varphi[\overline{m}] &&\text{(invariance with respect to basis)} \qquad \square
\end{aligned}$$

We have one situation in which we can recognize $\mathcal{L}_{\mathrm{t}}$-purity, theories T of topological modules which have complete elimination of set quantifiers from tppfs. "Elimination" here means that for every tppf φ of $\mathcal{L}_{\mathrm{t}}$, there is a ppf $\hat{\varphi}$ of ${}_R\mathcal{L}$ such that $T \models \varphi \leftrightarrow \hat{\varphi}$. Such theories include the theory of $\langle \mathbb{Q}, \mathcal{E} \rangle$ already discussed, and any theory with a compact model, cf. Kucera [17, Theorem 3.13]. For the latter sort of theory, $\hat{\varphi}$ is constructed explicitly from φ, and $\models_t \hat{\varphi} \to \varphi$ holds in all topological modules, so the result is a little sharper.

Theorem 6.7. *Suppose $\langle \mathcal{M}, \sigma \rangle$ is a subspace of $\langle \mathcal{N}, \tau \rangle$ such that $\mathcal{M}$ is a pure submodule of $\mathcal{N}$.*

If $\langle \mathcal{N}, \tau \rangle$ is $\mathcal{L}_{\mathrm{t}}$-equivalent to a compact module, or if $\langle \mathcal{M}, \sigma \rangle$ and $\langle \mathcal{N}, \tau \rangle$ are both models of an $\mathcal{L}_{\mathrm{t}}$-theory such that every tppf with only individual variables free is equivalent, modulo $\mathbf{T}$, *to a positive primitive formula, then $\langle \mathcal{M}, \sigma \rangle$ is $\mathcal{L}_{\mathrm{t}}$-pure in $\langle \mathcal{N}, \tau \rangle$.*

PROOF. We only need check that tppfs are reflected downwards. But this is easy: if φ is a topological positive primitive formula with only individual variables free, $\langle \mathcal{N}, \tau \rangle$ satisfies $\forall \bar{v}(\varphi \leftrightarrow \hat{\varphi})$, and in either case $\langle \mathcal{M}, \sigma \rangle$ satisfies $\forall \bar{v}(\hat{\varphi} \to \varphi)$. If $\overline{m} \in M$ and $\langle \mathcal{N}, \tau \rangle \models \varphi[\overline{m}]$, then $\langle \mathcal{N}, \tau \rangle \models \hat{\varphi}[\overline{m}]$, and in particular $\mathcal{N} \models \hat{\varphi}[\overline{m}]$ since $\hat{\varphi}$ is a formula of ${}_R\mathcal{L}$. Thus since $\mathcal{M}$ is pure in $\mathcal{N}$, $\mathcal{M} \models \hat{\varphi}[\overline{m}]$. Thus in either case $\langle \mathcal{M}, \sigma \rangle \models \hat{\varphi}[\overline{m}]$, and so $\langle \mathcal{M}, \sigma \rangle \models \varphi[\overline{m}]$. $\square$

So, for instance, $\langle \mathbb{Q}, \mathcal{E} \rangle$ is $\mathcal{L}_{\mathrm{t}}$-pure in $\langle \mathbb{R}, \mathcal{E} \rangle$. In fact, since $\mathbb{Q}$ is an injective abelian group, any embedding of $\mathbb{Q}$ onto a Euclidean subspace of a model of the $\mathcal{L}_{\mathrm{t}}$-theory of $\langle \mathbb{Q}, \mathcal{E} \rangle$ is $\mathcal{L}_{\mathrm{t}}$-pure in our sense. Since the two structures are in fact $\mathcal{L}_{\mathrm{t}}$-equivalent, it follows by the tpp elimination of quantifiers that this is an elementary embedding in the sense of Majewski [20].

Although this theorem allows us to recognize $\mathcal{L}_{\mathrm{t}}$-pure extensions in certain situations, it does not necessarily help to construct them. We could, for instance, reduce a set of tppfs with no set variables free to a set of equivalent pp formulas, and then use that set to construct a useful extension of some sort by the usual methods of the model theory of modules. However this method would not help to construct an appropriate topology on that extension. Indeed, Corollary 5.4 ensures that sometimes it will not be possible to do so, for a countable nondiscrete module *cannot* be a subspace of an $\aleph_1$-saturated (two-sorted) elementary extension.

7. The role of products and coproducts

Suppose that for each $i \in I$, $\mathcal{X}_i = \langle X_i, \tau_i \rangle$ is a topological space, and let $\mathbb{X} = \prod_{i \in I} X_i$. There are (at least) two topologies of interest on the Cartesian product: the *Tychonoff topology* τ_{prod} which has as a basis all sets of the form $\prod_{i \in I} U_i$, where $U_i \in \tau_i$ for all $i \in I$ and $U_i = X_i$ for all but finitely many $i \in I$; and the *box topology* τ_{box} which has as a basis all sets of the form $\prod_{i \in I} U_i$, where $U_i \in \tau_i$ for all $i \in I$. Clearly for infinite I, the box topology is strictly finer than the Tychonoff topology. It is a fundamental fact that the Tychonoff topology makes the Cartesian product into the direct product in the categorical sense, both in the category of topologies and in the categories of topological abelian groups or topological modules. Thus the projection maps are continuous (and open).

Coproducts in the category of topological spaces are just disjoint unions; but in the category of topological abelian groups the situation is quite different. Let $(\langle \mathcal{M}_i, \tau_i \rangle)_{i \in I}$ be a family of topological abelian groups (modules). The *coproduct topology* τ_{coprod} is defined to be the finest module topology on the direct sum $\bigoplus_{i \in I} \mathcal{M}_i$ which makes all the canonical inclusions $\varepsilon_i \colon \mathcal{M}_i \to \bigoplus_{i \in I} \mathcal{M}_i$ continous; since clearly both the restrictions from the direct product of the Tychonoff topology and of the box topology do so, the coproduct topology is well defined and at least as fine as the restriction of the box topology. (The set of Hausdorff module topologies on a fixed module $\mathcal{M}$ is a complete lattice.) In fact, the coproduct topology coincides with the box topology if I is countable; but it may be be finer if I is uncountable, cf. Nickolas [26]. It is also clear that for finite coproducts, the coproduct topology coincides with the Tychonoff topology. The basic ideas date back to the 1940s, cf. S. Kaplan [14], but the first complete (abstract) description of the coproduct topology was given by P. J. Higgins in 1977 [13], with the first explicit description being given by Chasco and Domínguez [5]; and Enns has a new and perhaps simpler way of deriving this description in his M.Sc. thesis [8].

Theorem 7.1 (Enns [8, Theorem 4.36]**).** *Let $(\langle \mathcal{M}_i, \tau_i \rangle)_{i=1}^n$ be topological modules and let $\varphi(\bar{v}, \overline{V})$ be a topological positive primitive formula. For each $i = 1, \dots, n$ let $\bar{a}^i$ be a tuple in M_i compatible with $\bar{v}$, and let $\overline{U}^i$ be a tuple in τ_i compatible with $\overline{V}$. Let $\mathbf{a}$ be the corresponding tuple formed pointwise in $\prod_{i=1}^n M_i$ and let $\mathbf{U}$ be the tuple in the product topology with components $(\overline{U}^1)_j \times \cdots \times (\overline{U}^n)_j$. (So each such component is a member of the standard basis for the product topology).*

Then

$$\prod_{i=1}^{n} \langle \mathcal{M}_i, \tau_i \rangle \models \varphi[\mathbf{a}, \mathbf{U}]$$

if and only if

$$\langle \mathcal{M}_i, \tau_i \rangle \models \varphi[\bar{a}^i, \overline{U}^i] \quad \text{for each } i,\ i = 1, \dots, n.$$

Proof. The proof is a routine induction on the complexity of topological positive primitive formulas. □

The "correct" generalization of this theorem would be to infinite products; or perhaps to infinite coproducts. But neither of these can work. The specific form of $\mathbf{U}$ is required for the induction to carry through; and in the infinite case the sets of this form are precisely a base for the box topology. So the theorem may fail for infinite products, since in general the sets $\prod_{i \in I} \overline{U}^i{}_j$ are not open in the product topology; and the theorem may fail for infinite coproducts since in general the sets $\prod_{i \in I} \overline{U}^i{}_j$ do not form a base for the coproduct topology. For what it is worth, this theorem does carry through for arbitrary products when endowed with the box product topology; and thus by the result [26] of Nickolas already cited, for countable direct sums as well.

Flum and Ziegler [9] develop a general theory of back-and-forth arguments for $\mathcal{L}_{\text{t}}$; and using these methods in particular prove the following theorem:

Theorem 7.2 ([9, Part I, Theorem 6.1])**.** *Suppose that $\langle \mathcal{M}_i, \sigma_i\rangle$ and $\langle \mathcal{N}_i, \tau_i\rangle$ are $\mathcal{L}_t$-elementarily equivalent $\aleph_0$-saturated topological R-modules for each $i \in I$. Then*

$$\prod_{i\in I}\langle \mathcal{M}_i, \sigma_i\rangle \equiv_t \prod_{i\in I}\langle \mathcal{N}_i, \tau_i\rangle$$

We may remove the requirement that the modules be $\aleph_0$-saturated if the language is finite, for instance in the case of topological abelian groups.

It is not known if this result holds in complete generality.

By similar methods, Enns proved in his thesis:

Theorem 7.3 ([8, Theorem 4.38])**.** *Suppose that $\langle \mathcal{M}_i, \sigma_i\rangle$ and $\langle \mathcal{N}_i, \tau_i\rangle$ are $\mathcal{L}_t$-elementarily equivalent $\aleph_0$-saturated topological R-modules for each $i \in I$. Then*

$$\left\langle \bigoplus_{i\in I} \mathcal{M}_i, \sigma_{\text{coprod}}\right\rangle \equiv_t \left\langle \bigoplus_{i\in I} \mathcal{N}_i, \tau_{\text{coprod}}\right\rangle.$$

Once again, we may remove the requirement that the modules be $\aleph_0$-saturated for the case of finite languages, in particular for topological abelian groups; and we do not know if the result holds in complete generality.

Finally we consider the relationship between the direct sum and the direct product. We get a result, but not a direct generalization of the corresponding Theorem 1.5 for ordinary modules. This result depends on an appropriate choice of basis in order to allow the use of set parameters.

The proof consists almost entirely in choosing an efficient notation to describe the various elements and sets involved. Let $(\langle \mathcal{M}_i, \sigma_i\rangle)_{i\in I}$ be topological modules. Let $\mathcal{M}_\oplus = \bigoplus_{i\in I}\mathcal{M}_i$; let $\mathcal{M}_\Pi = \prod_{i\in I}\mathcal{M}_i$. Let $\bar{a} \in \mathcal{M}_\oplus$, $\overline{U} \in \beta$, where β is the standard basis for the Tychonoff product topology. The standard projection and inclusion maps are π_i and ε_i respectively. The *support* of $a \in \mathcal{M}_\oplus$ is $\text{supp}(a) = \{i \in I : \pi_i(a) \neq 0\}$; the *support* of $U \in \beta$ is $\text{supp}(U) = \{i \in I : \pi_i[U] \neq M_i\}$; and the *support* of $\bar{a}\overline{U}$, $\text{supp}(\bar{a}\overline{U})$, is the union of the supports of the elements in $\bar{a}$ and the sets in $\overline{U}$. Clearly $\text{supp}(\bar{a}\overline{U})$ is a finite subset of I. Given $I_0 \subseteq I$ and $c \in \mathcal{M}_\Pi$, let $c \restriction I_0$ be the element d defined by $d_i = c_i$ if $i \in I_0$; otherwise let $d_i = 0$. Clearly if I_0 is finite, then $d \in \mathcal{M}_\oplus$. If $\bar{c} = c^0, c^1, \ldots$ is a tuple in $\mathcal{M}_\Pi$, let $\bar{c} \restriction I_0 = c^0 \restriction I_0, c^1 \restriction I_0, \ldots$.

Lemma 7.4 (Enns [8, Lemma 4.39]**).** *Let $\psi(\bar{v}, \bar{u}, \overline{V})$ be a topological positive primitive formula, and let all other notation be as in the preceding paragraph. Let $I_0 = \text{supp}(\bar{a}\overline{U})$. Then*

$$\langle \mathcal{M}_\Pi, \tau_{\text{prod}}\rangle \models \psi[\bar{a}, \bar{c}, \overline{U}] \implies \langle \mathcal{M}_\Pi, \tau_{\text{prod}}\rangle \models \psi[\bar{a}, \bar{c} \restriction I_0, \overline{U}].$$

Proof. Obvious. Note that in computing the value of a term, which is done component-wise in the product, for i outside the support of $\bar{a}\overline{U}$ the value only depends on $\bar{c}$, not on $\bar{a}$, and membership is trivial because the corresponding open set is all of M_i. Therefore replacing the i-components of the entries in $\bar{c}$ by 0 cannot affect the truth of atomic formulas, and similarly for conjunctions of such. Then the result follows easily by induction on the quantifier complexity of the topological positive primitive formula ψ. The case of universal set quantifiers is immediate since the hypothesis is taken over a basis for the product topology; and the case for existential individual quantifiers follows easily. □

Theorem 7.5 (Enns [8, Theorem 4.40]**).** *Continuing the notation of the lemma and the paragraph preceding it, let $\psi(\bar{v}, \overline{V})$ be a topological positive primitive formula. Then*

$$\langle \mathcal{M}_\oplus, \tau_{\mathrm{prod}} \restriction \mathcal{M}_\oplus \rangle \models \psi[\bar{a}, \overline{U} \cap \mathcal{M}_\oplus] \iff \langle \mathcal{M}_\Pi, \tau_{\mathrm{prod}} \rangle \models \psi[\bar{a}, \overline{U}].$$

Proof. The forward direction follows by Proposition 6.1 since we have put the subspace topology on $\mathcal{M}_\oplus$. The converse follows immediately by Lemma 7.4. □

Corollary 7.6. *With the same notational conventions, $\langle \mathcal{M}_\oplus, \tau_{\mathrm{prod}} \restriction \mathcal{M}_\oplus \rangle$ is $\mathcal{L}_{\mathrm{t}}$-pure in $\langle \mathcal{M}_\Pi, \tau_{\mathrm{prod}} \rangle$.*

Note that we do not and cannot get a nice relation between the coproduct and the product. For an arbitrary coproduct of discrete modules is discrete, but the corresponding product is not.

8. Pure-injectivity

We seek some suitable definition of pure-injectivity for topological modules in the form of "injective over topologically pure embeddings." We have already discussed a range of candidates for "topological pure embedding," with the two alternatives of Definition 6.5 perhaps showing the most promise.

Immediately we see another obstruction to the use of free set variables in tpp formulas:

Proposition 8.1 (Enns [8, Corollary 4.27]**).** *Let $\varphi(\bar{v}, \overline{V})$ be a topological positive primitive formula. Form $\varphi^\circ(\bar{v})$ from φ by deleting every conjunct of the open matrix of φ of the form "$\mathbf{t} \in V$", V a free set variable of φ. Then φ is satisfiable in $\langle \mathcal{N}, \tau \rangle$ if an only if for some $\bar{a}$ in N, $\langle \mathcal{N}, \tau \rangle \models \varphi[\bar{a}, \overline{M}]$, equivalently $\langle \mathcal{N}, \tau \rangle \models \varphi^\circ[\bar{a}]$.*

Proof. All occurrences of set variables in a topological positive primitive formula are positive occurrences of some atomic subformula "$\mathbf{t} \in V$." So if such a formula is true at a given assignment of values, it remains true upon replacing the value of V by any larger open set. □

Thus the satisfiability of a set of topological positive primitive formulas does not depend on the free set variables appearing in that set.

In the end, it appears not to matter much what the precise forms of the definitions are: there are strong barriers to proving either direction of the equivalence "topological pure-injective if and only if tpp-compact"; at least by anything approaching the standard methods of the model theory of modules.

8.1. Pure injective implies tpp-compact? First consider the case where $\langle \mathcal{N}, \tau \rangle$ is injective over topologically pure maps in some sense. The "short" proof that pure-injective implies pp-compact for ordinary modules uses quite heavy machinery, namely the existence of sufficiently saturated elementary extensions. Reduced products in two-sorted logic can be used just as they are for ordinary modules, but the embedding constructed will in general be onto a discrete subspace, and therefore not even continuous. So we cannot carry through the proof of Section 1 for $\mathcal{L}_{\mathrm{t}}$ pure embeddings or any similar concept that demands the topologies of the two spaces be related in some way.

For injectivity over weakly $\mathcal{L}_{\mathrm{t}}$-pure embeddings, we can imitate the proof of Section 1 as long as we only demand that the homomorphisms we consider preserve topological positive primitive formulas; there is nothing that we can use to force the lifted homomorphism $\hat{f}$ to be continuous. In particular, although continuity of definable functions within a structure is axiomatizable in $\mathcal{L}_{\mathrm{t}}$, we cannot use formulas or sets of formulas of $\mathcal{L}_{\mathrm{t}}$ to force a homomorphism between structures to be continuous.

8.2. tpp-compact implies pure-injective? Now we consider whether various forms of tpp-compactness might imply some form of injectivity over pure maps for topological modules. So suppose that $\langle \mathcal{N}, \tau \rangle$ is tpp-compact, that $e \colon \langle \mathcal{A}, \alpha \rangle \to \langle \mathcal{B}, \beta \rangle$ is a pure embedding, and that $f \colon \mathcal{A} \to \mathcal{N}$ is a homomorphism. At the very least we assume that e reflects tppfs and that f preserves tppfs (with no set variables free). Then we can imitate the usual proof for ordinary modules: we enumerate B by a set of new, distinct variables $\mathcal{V}$, and let Φ be the set of all tppfs in these variables and with parameters from A satisfied by B. Since e reflects tpp formulas, Φ is finitely satisfiable in $\mathcal{A}$, and since f preserves tpp formulas, the natural image $f[\Phi]$ of Φ in $\mathcal{N}$ is also finitely satisfiable. So if $\langle \mathcal{N}, \tau \rangle$ is compact with respect to sets of tpp formulas with no free variables, $f[\Phi]$ has a solution in $\langle \mathcal{N}, \tau \rangle$ and so there is a map $\hat{f} \colon \langle \mathcal{B}, \beta \rangle \to \langle \mathcal{N}, \tau \rangle$ induced by this solution. Clearly by construction $\hat{f}$ preserves tpp formulas, so in particular $\hat{f}$ is a homomorphism. But that is the only condition that we can impose on $\hat{f}$ by tpp formulas, even if we assume that e is a topological embedding and that f is continuous. There is no sense in which the topology on $\mathcal{B}$ is determined by Φ.

Ignoring for the moment the difficulties already noted in including set parameters anywhere in our definitions, we might ask what could be gained by allowing set parameters. In order to carry set variables through the above argument, it is necessary to assume right from the start that e is a topological embedding and that f is proper. Then $\hat{f}$ would be constructed as a two sorted map $\hat{f}_0 \colon \mathcal{B} \to \mathcal{N}$ and $\hat{f}_1 \colon \beta \to \tau$. Since $\hat{f}$ preserves tpp formulas, in particular "$v \in V$," we obtain immediately that for all nonempty $U \in \beta$, $\hat{f}_0[U] \subseteq \hat{f}_1(U)$, that is, $\hat{f}_0[U] = \hat{f}_1(U) \cap \hat{f}_0[M]$. So $\hat{f}_0$ is open onto its image. But we still have no tools available to conclude that $\hat{f}_0$ is continuous.

9. Alternative approaches

In [2] Paul Bankston introduced a notion of ultraproduct for topological spaces; and has expanded on this idea and used techniques associated with it in subsequent papers over the years. This gives us a sort of model theory of topological spaces without the bother of considering any sort of formal language.

In 1976, C. Ward Henson [12] introduced a logic suitable for the study of Banach spaces. In addition to the language of $\mathbb{Q}$-vector spaces, the appropriate language includes predicates for the closed unit ball and for the elements of norm greater than or equal to one. Notions of "approximate satisfaction" (of positive bounded formulas) and "approximate equivalence" grow naturally out of the metric structure of the Banach spaces. This line of study has proved to be very profitable in the study of Banach spaces, but this sort of logic has little to say about topological modules in general. Closely related to this is the *continuous logic* of Ben Yaacov, Henson, et al., cf. [4] which gives a good model theory for structures endowed with a metric.

Pillay [27] studied topological structures $\langle \mathcal{M}, \sigma \rangle$ where the topology is uniformly definable in some first-order language, that is, there is a formula $\beta(v, \bar{u})$ of the language of $\mathcal{M}$ such that $\{\beta[M, \overline{m}] : \overline{m} \in M\}$ is a basis for σ. Many of the structures we consider here do not fall into this class. Pillay obtains especially nice results for structures where every definable set is a Boolean combination of definable open sets.

There is a rich history of the study of the class of locally compact abelian groups (LCA), motivated by the Pontryagin duality theorm. In 1967 Moskowitz [25] considered to what extent homological algebra can be developed inside LCA. As we have already noted, this category is not abelian, so there are restrictions on what can be accomplished. Moskowitz obtains general results for LCA, but often by restricting his attention to cases where the maps are proper and where one or more of the abelian groups in questions is compactly generated. This opens up the possibility of imitating the development of the model theory of modules in a functorial context, as in Prest [29], with the same sorts of restrictions as considered by Moskowitz. This would *not* need references to the syntax and semantics of any particular language.

We have already made reference to the recent work of Peter Loth and others on definitions of purity and pure-injectivity in LCA. Khan [15] first introduced a concept of purity for topogical abelian groups in 1973; the book of Armacost [1, Chapter 7] of 1981 contains this and much else that is relevant. More recently, Peter Loth has reconsidered the definition of Khan and several of his own; these definitions of purity in LCA are all distinct. See [18, 19] for some of Loth's work most closely connected to the ideas that we explore here, and for further references.

Definition 9.1 (Khan [15]). Let $\langle \mathcal{A}, \sigma \rangle$ be a locally compact abelian group. A closed subgroup H of A is *t-pure* in $\langle \mathcal{A}, \sigma \rangle$ if for all closed subgroups $K \supseteq H$ such that K/H is compactly generated, H splits from K.

Theorem 9.2 (Khan [15]). *Let $\langle \mathcal{A}, \sigma \rangle$ be a locally compact abelian group and H a closed subgroup of $\langle \mathcal{A}, \sigma \rangle$.*

(1) *If H is a topological direct summand of $\langle \mathcal{A}, \sigma \rangle$ then H is t-pure in $\langle \mathcal{A}, \sigma \rangle$.*
(2) *If H is t-pure in $\langle \mathcal{A}, \sigma \rangle$ then H is pure in $\langle \mathcal{A}, \sigma \rangle$.*

Thus "t-purity" as defined by Khan extends algebraic purity for abelian groups and shares the important property (1) with it. There appears to be no easy correspondence between the notions of purity relative to $\mathcal{L}_{\mathrm{t}}$ that we have introduced, and Khan's notion of t-purity.

Definition 9.3 (Loth [19]). A proper short exact sequence

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

of locally compact abelian groups is *t-pure* if $\alpha[A]$ is a *topologically pure subgroup* of B, that is, if

$$\mathbf{cl}_B(n\alpha[A]) = \mathbf{cl}_B(nB) \cap \alpha[A]$$

for every positive integer n.

By taking $n = 1$ it follows that A must be closed in B.

Definition 9.4 (Loth [19]). A proper short exact sequence

$$0 \to A \to B \to C \to 0$$

of locally compact abelian groups is *topologically pure* if

$$0 \to \mathbf{cl}_A(nA) \to \mathbf{cl}_B(nB) \to \mathbf{cl}_C(nC) \to 0$$

is proper exact for all positive integers n.

Since the sequences considered are proper, the image of A is locally compact and hence closed in B. Thus a topologically pure exact sequence is t-pure. But Loth [18, Example 3.5] gives an example of a t-pure exact sequence which is not topologically pure. In addition ([18, Example 2.4]) shows that a proper exact sequence that is algebraically pure need not be topologically pure, and that a proper exact sequence which is topologically pure need not be algebraically pure. It follows in particular by Theorem 9.2 that neither of Loth's concepts imply t-purity in the sense of Khan.

In regards to topological purity, the sets nA are pp-definable subgroups; and we have already noted that the closure of a tpp-definable subgroup is again tpp-definable (see the discussion preceding Proposition 6.4). Thus "topological purity" demands only that a restricted class of tpp formulas be reflected.

Loth [19] defines *t-pure-injectivity* in the class of locally compact abelian groups in the usual way from (his version of) t-purity. In this paper, Theorem 2.7 shows that the t-pure-injective locally compact abelian groups have a rather restricted structure: they are a topological direct sum $\mathbb{R}^n \oplus \mathbb{T}^\kappa \oplus G$, where n is a nonnegative integer, κ is a cardinal, and G is a"topological torsion group." We refer the reader to Loth's papers for the details.

10. Summary

We have considered the possibility of developing a useful model theory of topological modules (useful for the purposes of topological algebra) in the languages $\mathcal{L}_\mathrm{t}$ or $\mathcal{L}_\mathrm{m}$.

- Any approach to this problem that involves the use of open-set parameters, or imposes topological constraints on the morphisms between structures, seems unlikely to produce anything useful.
- If we consider the category of topological modules with module homomorphisms that preserve topological positive primitive formulas, we can get some results of a model-theoretic nature, but which appear to have little of topological interest.
- Nothing we have discussed here places any limits on what might be accomplished by other means. In particular, it places no restrictions on what might be accomplished within the (nonaxiomatizable) class of locally compact abelian groups or modules.

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